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On the nonlinear stability of the oscillatory viscous flow of an incompressible fluid in a curved pipe

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The stability of the flow of an incompressible, viscous fluid through a pipe of circular cross-section, curved about a central axis is investigated in a weakly nonlinear regime. A sinusoidal pressure gradient with zero mean is imposed, acting along the pipe. A WKBJ perturbation solution is constructed, taking into account the need for an inner solution in the vicinity of the outer bend, which is obtained by identifying the saddle point of the Taylor number in the complex plane of the cross-sectional angle coordinate. The equation governing the nonlinear evolution of the leading-order vortex amplitude is thus determined. The stability analysis of this flow to axially periodic disturbances leads to a partial differential system dependent on three variables, and since the differential operators in this system are periodic in time, Floquet theory may be applied to reduce it to a coupled infinite system of ordinary differential equations, together with homogeneous uncoupled boundary conditions. The eigenvalues of this system are calculated numerically to predict a critical Taylor number consistent with the analysis of Papageorgiou (1987). A discussion of how nonlinear effects alter the linear stability analysis is also given. It is found that solutions to the leading-order vortex amplitude equation bifurcate subcritically from the eigenvalues of the linear problem.

1. Introduction

Our concern is with the stability of the unsteady viscous flow of an incompressible fluid in a pipe of circular cross-section, itself curved about a central axis, and subject to a sinusoidal pressure gradient of zero mean. In particular, we are interested in the effects of the nonlinear terms of the Navier–Stokes equations on the development of the disturbance at the pipe walls. The stability of laminar periodic flows, such as in the problem under consideration, is often of both mathematical and physical importance. In this case, our model is of particular relevance to the fluid mechanics of the cardiovascular system, and in particular the possible links to atheroma. Atheroma is increasingly an important disease process in middle age, and indeed postmortem results even in young people have shown evidence of changes to large arterial vessels. In the initial stages fatty streaks – accumulation of lipid in the tunica intima – occur. Subsequently, raised plaques become visible on the surface of the arteries. An important feature of atheroma is that it develops at preferred sites in the circulatory system. Early signs may be seen in the major central arteries, and not in the peripheral system, and indeed the later onset of atheroma in the peripheral circulation might well be due to more advanced lesions in larger vessels influencing the blood flow past them and hence the development downstream. The first sites tend to be found near junctions and curves in the arteries. For example, the posterior wall of the descending thoracic aorta almost always has fatty streaks, whilst the anterior wall does not. It is thought that one or both of two factors are important in the preferential distribution: first the structure of the vessel wall varies so that some sites are more prone to development, and secondly that they are subject to different influences due to the fluid flow, and this might alter the physiological or biochemical processes taking place. Clearly we are interested in the latter. Experimental evidence indicates that atheroma is linked with low permeability of the endothelium which inhibits the efflux of cholesterol. It was initially thought that this low permeability might be due to damage to the endothelium produced by high wall shear stresses, but in fact it has been shown that such damage increases the rate at which molecules may diffuse into and out of the plasma. Thus we would expect that less streaks are visible at regions of high shear, which agrees with observational evidence. For example, in junctions streaks are observed on the outer walls, but not on the flow divider, and in curved arteries evidence of atheroma is more prevalent on the inner bends. For a more detailed account of atherogenesis and references to experimental work the reader is referred to Caro *et al.* (1978).

Clearly, the model used to study the fluid flow in a curved pipe is far simpler than the blood flow within the aortic arch. For example, our model does not take into account the distensibility of the arterial wall. However, although this would be of importance in determining the local pressure gradient it does not affect the overall velocity distribution, since the wavelength of the pulse of fluid driven by the pressure gradient is far greater than the distance travelled by a typical fluid element in a single cycle. For example, in the canine aorta, the pulse wavelength is 3–4 m, whilst the distance travelled by the fluid element is about 10–20 cm, and hence we can assume that during one period the cross-sectional area of the section of the pipe traversed by the fluid is approximately uniform. Similarly elastic effects can also be neglected provided that there is no discernible change in the pipe cross-section over a length scale of say 10 cm.

Several problems dealing with the stability of oscillatory flows have been investigated, both theoretically and experimentally (see Davis 1976). Periodic laminar flows may be categorized in two ways. First those which have a non-zero mean velocity, in which case the disturbance is usually associated with the mean flow and the parameters governing the stability problem are dependent on the unperturbed flow field. Secondly, periodic flows may be purely oscillatory, that is have zero mean. In general perturbation methods, although often applicable to flows of non-zero mean, cannot be employed to study such disturbances and numerical techniques must be relied upon to analyse the stability problem. The flow field investigated in this paper is purely oscillatory in nature and thus falls into the latter of these groups.

Since viscosity is to be included, of importance are the stability mechanisms operating within the Stokes layers formed at the pipe walls, and in particular those associated with specific types of geometry and local surface behaviour. Since we are investigating the flow within a curved pipe, clearly centrifugal effects are of importance, and will affect the development of the instability differently depending on whether the surface is locally convex or concave.

Experimental and theoretical investigations of the centrifugal instabilities of a Stokes layer in time-periodic flows were done by Seminara & Hall (1976). They investigated the linear stability of the flow induced by a cylinder oscillating harmonically about its axis in an unbounded fluid. Within the Stokes layer at the surface of the body, the flow is shown to be unstable to a Taylor-vortex-like mode of instability, with the vortices being periodic in the azimuthal direction for sufficiently high frequencies of oscillation. Also considered was the possible relevance of their study to the stability of the flow within the aortic arch. A description of the flow here is complicated, but less important features of the problem were neglected. Previous work by Lyne (1971), had shown that, in the high-frequency limit, viscous effects are confined to a thin layer at the wall, suggesting a Stokes-layer-type flow regime there. Seminara & Hall also investigated the flow induced by an oscillating curved pipe, and in particular the stability characteristics of the inner (convex) and the outer (concave) bend. They found that at the inner bend the disturbance is locally unstable, whilst at the outer bend it is locally stable. If, however, the flow is driven by an oscillating pressure gradient, as is the case for our problem, the stable and unstable regions change positions. A more detailed experimental treatment of this problem was carried out by Park, Barenghi & Donnelly (1980) and confirmed the secondary subharmonic destabilization of the most dangerous mode found by Seminara & Hall.

Hall (1984) and Papageorgiou (1987) found that the type of instability mechanism investigated by Seminara & Hall can also occur at spatially localized positions in more complex boundary layer flows.

Hall (1984) considered the instability of the two-dimensional flow induced by a transversely oscillating cylinder, of both elliptical and circular cross-section, in an unbounded viscous fluid, in both the linear and weakly nonlinear regimes. For frequencies sufficiently high, the cylinder motion drives an unsteady boundary layer which is unstable to Taylor–Görtler vortices, localized in regions where the slip velocity of the potential flow, outside the boundary layer, is parallel to the motion. For the circular cylinder case in the weakly nonlinear regime, it was found that the finite-amplitude solutions to the evolution equation for the leading-order vortex amplitude bifurcate subcritically from the eigenvalues of the linear problem.

In the problems outlined above, the flow field was essentially two-dimensional. However the problem under consideration in this paper comprises the basic motion in the pipe core, together with a small secondary two-dimensional flow in the cross-section, and thus the underlying flow is no longer two-dimensional. The steady problem was first studied by Dean (1927, 1928), who concluded that the motion depends on a parameter, K, defined by

$$K = 2R_e^2\frac{a}{R},$$

where a is the pipe radius, R its radius of curvature about some central axis, and R_e the Reynolds number. Dean's analysis was restricted to small values of K, but subsequently has been extended numerically to cover moderately large K, and asymptotic boundary layer theory has been used to obtain results for very large values of this parameter.

Until Lyne (1971), the time-dependent problem had not been studied. He gave a detailed asymptotic analysis of the fully developed unperturbed flow in a curved pipe under the action of a pressure gradient assumed to be sinusoidal in time, and with zero mean. It is also assumed that δ, defined as

$$\delta = \frac{a}{R},$$

is small. The flow is found to depend on two parameters,

$$\epsilon^2 = \left(\frac{W^2}{a\omega^2}\right)\frac{1}{R}, \quad R_s = \frac{W^2 a}{R\omega\nu},$$

where ν is the kinematic viscosity, W is a typical velocity along the pipe, and ω is the frequency of oscillation of the basic flow. The parameter ϵ^2 is the ratio of the square of the displacement amplitude of the axial motion, $(W^2/a\omega^2)$, to the radius of curvature, R, and is taken to be small. R_s may be thought of as the Reynolds number of the secondary flow. Also of importance in the analysis is the parameter β, given by,

$$\beta^2 = \left(\frac{2\nu}{\omega}\right)\frac{1}{a^2} = \frac{2\epsilon^2}{R_s};$$

β is also assumed to be small. Physically, β represents the ratio of the Stokes layer thickness, $(2\nu/\omega)^{1/2}$, to the pipe radius, a. Since β is small, the viscous effects are confined to a thin layer on the pipe wall, whilst the flow in the pipe core is inviscid. The influence of the parameter β on physiological flows was first recognized by Womersley (1955).

The linear stability problem was investigated theoretically by Papageorgiou (1987) and a more detailed discussion of the results obtained is given in §2.2, since this provides a basis for the nonlinear problem under consideration here.

Since the Stokes layer is thin, the streamlines adjacent to the wall can be assumed to have radius of curvature of $O(R)$, and we introduce, T_a, the Taylor number for the secondary flow and defined by

$$T_a = \frac{W^2\left((2\nu/\omega)^{1/2}\right)^3}{R\nu^2} = 2\frac{W^2}{R\omega}\frac{a}{\nu}\left(\frac{2\nu}{a^2\omega}\right)^{1/2} = 2R_s\beta.$$

Since we are interested in centrifugal effects, we demand that T_a is $O(1)$, and choose R_s to be

$$R_s = 2T\beta^{-1},$$

where $T_a = 4T$. Note that we may relate the parameters, ϵ, β, T through the formula

$$\epsilon^2 = \beta T,$$

and thus the problem may be reduced to one dependent only on the parameters β and T. Since T_a is much smaller than the Taylor number for the axial flow, we may assume that the vortices will be aligned with the flow down the pipe and will have characteristic wavenumbers based on the Stokes layer thickness, $(2\nu/\omega)^{1/2}$. In the construction of the solution two length scales, $O(1)$ and $O(\beta)$, emerge as being of importance. A possible approach, taking into account the different scalings in the pipe cross-section, would be to expand the perturbation quantities in powers of β and apply a WKBJ method. Walton (1978) adopted such a method when investigating the narrow spherical annulus problem. He anticipated that the critical Taylor number at the equator, above which instabilities in the flow may occur, was slightly in excess of that for the corresponding cylinder problem. It was found that the solution becomes singular in the vicinity of the equator, suggesting that an inner expansion is required to smooth out this singularity. The solutions to the resulting amplitude equation could be expressed in terms of Airy functions, which have the property that solutions decaying at infinity exhibit oscillatory behaviour at minus infinity. Clearly such functions are physically unacceptable, since we require that solutions for the velocity distribution

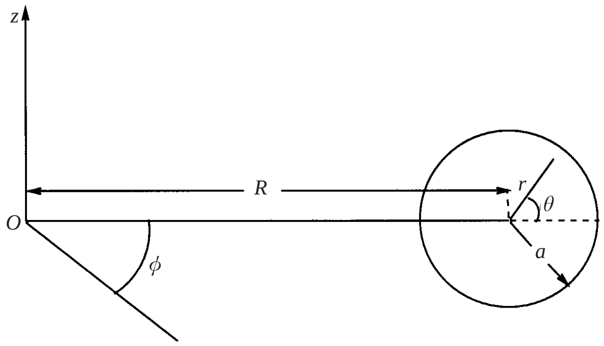

FIGURE 1. The coordinate system.

must be symmetric about the equator. The difficulty was resolved by Soward & Jones (1983), who identified the Taylor number for which inner solutions are well behaved away from the equator.

For our problem, the linear stability analysis of Papageorgiou showed that, as for the narrow spherical annulus problem, the solution to the amplitude equation was singular, in this case in the neighbourhood of the outer bend. Again a local inner expansion produced physically unacceptable results and the method of Soward & Jones (1983), was employed to identify the correct Taylor number for which the inner solution is well behaved at $\pm\infty$. Papageorgiou's investigation also indicated that the instability sets in first at the outer bend of the pipe, and it was also shown that there exists no critical Taylor number for the corresponding problem at the inner bend, suggesting that centrifugal effects are of little importance here.

The procedure for the remainder of this paper is as follows. In the coming section we formulate the problem at hand, obtaining the governing equations for the flow field within the Stokes layer. In § 2.2 we present a brief summary of the derivation of the linear amplitude equation derived by Papageorgiou. This analysis is then extended in § 3 to include the nonlinear terms of the governing equations and thus the evolution equation for the leading-order vortex amplitude is obtained in this weakly nonlinear regime. In § 4 numerical results for both the linear and nonlinear problems are presented and discussed. Finally we draw some conclusions.

2. Formulation of the problem

2.1. *Equations of motion*

Consider the flow of an incompressible viscous fluid in a pipe of circular cross-section and radius a, which itself is curved in a circle of radius R about a central axis, as illustrated in figure 1. The spatial coordinates are taken to be (r, θ, ϕ) where r and θ are polar coordinates within the pipe cross-section, and $R\phi$ measures the distance along the pipe. The velocity vector $\boldsymbol{u}$ has components (u, v, w) corresponding to the (r, θ, ϕ) coordinate system and is assumed to be independent of ϕ, once the basic flow is fully developed.

A sinusoidal pressure gradient of the form

$$-\frac{\partial}{\partial \phi}\left(\frac{p}{\rho}\right) = RW\omega\cos(\omega t)$$

is imposed on the above flow regime. If we consider the Navier–Stokes equations for the system, and in particular the balance between the viscous work and pressure terms, we see that there exists a layer of thickness $O(2\nu/\omega)^{1/2}$ on the pipe walls inside which the viscous terms dominate, whilst in the core we have a potential flow. Similar analysis for the r- and θ-momentum equations suggests that within the viscous layer u and v are $O(W^2\beta/R\omega)$ and $O(W^2/R\omega)$ respectively. Thus we introduce the following non-dimensional notation:

$$\left.\begin{array}{lll} u' = \dfrac{u}{W^2/R\omega}, & v' = \dfrac{v}{W^2/R\omega}, & w' = \dfrac{w}{W}, \\ p' = \dfrac{p}{\rho(a/R)W^2}, & r' = \dfrac{r}{a}, & \tau = \omega t. \end{array}\right\} \tag{2.1}$$

Substituting (2.1) into the Navier–Stokes equations yields

$$u_r + \epsilon^2\left(uu_r + \tfrac{1}{r}vu_\theta - \tfrac{1}{r}v^2\right) - w^2\cos\theta = -p_r - \tfrac{1}{2}\beta^2\tfrac{1}{r}\tfrac{\partial}{\partial\theta}\left(v_r + \tfrac{1}{r}v - \tfrac{1}{r}u_\theta\right), \tag{2.2a}$$

$$v_r + \epsilon^2\left(uv_r + \tfrac{1}{r}vv_\theta - \tfrac{1}{r}uv\right) + w^2\sin\theta = -\tfrac{1}{r}p_\theta + \tfrac{1}{2}\beta^2\tfrac{1}{r}\tfrac{\partial}{\partial r}\left(v_r + \tfrac{1}{r}v - \tfrac{1}{r}u_\theta\right), \tag{2.2b}$$

$$w_r + \epsilon^2\left(uw_r + \tfrac{1}{r}vw_\theta\right) = \cos\tau + \tfrac{1}{2}\beta^2\left(w_{rr} + \tfrac{1}{r}w_r + \tfrac{1}{r^2}w_{\theta\theta}\right), \tag{2.2c}$$

$$u_r + \tfrac{1}{r}u + \tfrac{1}{r}v_\theta = 0. \tag{2.2d}$$

For ease in notation, we have ignored the superscripts on the new, non-dimensional variables. The system of equations above, (2.2*a*–*d*), describes the viscous flow field both near the wall of the pipe and in its core. The continuity equation of (2.2*d*) may be satisfied by introducing a non-dimensional stream function ψ, with u and v given by

$$u - \frac{1}{r}\psi_\theta, \quad v = -\psi_r.$$

We are primarily interested in the stability of the flow field adjacent to the walls, i.e. within the Stokes layer, which is of thickness, $(2\nu/\omega)^{1/2}\beta$. Thus we introduce the following scalings for r and ψ:

$$\eta = \beta^{-1}(1-r), \quad \Psi = \beta^{-1}\psi,$$

where η and Ψ are the new radial coordinate and stream function respectively, inside the Stokes layer. The solution for the basic flow within this layer is given by Lyne as a series expansion in β,

$$\Psi = \Psi_0 + \beta\Psi_1 + \beta^2\Psi_2 + \cdots, \tag{2.3a}$$

$$w = w_{B0} + \beta w_{B1} + \beta^2 w_{B2} + \cdots \tag{2.3b}$$

In (2.3*a*,*b*), Ψ_i and w_{Bi} for $i = 0, 1, 2\ldots$, are functions of τ, η and θ and are found for fixed values of R_s, the secondary Reynolds number. Expressions for Ψ_0, Ψ_1 may be found in Lyne (1971). As was stated in the Introduction, we require the asymptotic solution, as $\beta \to 0$ and $R_s \to \infty$.

The first two terms in the expansion for w_B are given by

$$w_{B0} = \sin\tau - \mathrm{e}^{-\eta}\sin(\tau - \eta), \tag{2.4a}$$

$$w_{B1} = -\tfrac{1}{2}\eta\mathrm{e}^{-\eta}\sin(\tau - \eta). \tag{2.4b}$$

Denoting the basic flow by $\boldsymbol{U}_B = (u_B, v_B, w_B, p_B)$, we have the following expressions for u_B and v_B:

$$u_B = \beta u_{B0} + \beta^2 u_{B1} + \cdots, \quad v_B = v_{B0} + \beta v_{B1} + \cdots, \tag{2.5a,b}$$

where

$$u_{Bi} = \frac{\partial \Psi_i}{\partial \theta}, \quad v_{Bi} = \frac{\partial \Psi_i}{\partial \eta} \tag{2.6}$$

for $i = 0, 1 \ldots$.

The solutions (2.5*a*,*b*) are strictly valid only in the limit as $R_s \to \infty$ with β held fixed. However, since R_s has no explicit effect on the equations for Ψ_0, w_{B0} etc. within the Stokes layer, but only affects the core flow, where it acts like the conventional Reynolds number, the solutions above, (2.4*a*,*b*), (2.5*a*,*b*) are indeed valid representations of the flow field as $\beta \to 0$.

Before proceeding with the analysis, we re-write the Navier–Stokes equations (2.2*a*–*d*) in terms of the Stokes variable η and thus obtain the system of equations governing the flow field within the Stokes layer.

2.2. *Review of the linear problem*

We now present a brief summary of the inner analysis and the results obtained by Papageorgiou (1987). For a more complete description of the analysis and the techniques employed, the reader is referred to this paper.

In the Introduction we mentioned that, for both the narrow spherical annulus near the equator and the curved pipe problem in the vicinity of the outer bend, inner expansions lead to physically unrealistic results. Such problems can be resolved by the analytic continuation of the solution into the complex θ-plane. Soward & Jones first used such a method to resolve the narrow spherical annulus problem, and Papageorgiou modified their method when considering the linear stability of the flow in a curved pipe. We now describe this approach when applied to the study undertaken here. The reader is referred to Soward & Jones (1983), and Heading (1962) for a more complete discussion of the formation of the inner solution in these cases.

We wish to obtain a dispersion relation for the Taylor number T of the form

$$T = T(k, \theta, \sigma), \tag{2.7}$$

where σ is the growth rate, and k is a wavenumber. T is an analytic function of the complex variables k, θ, and σ, but we require that, for physical flows, it remain real and constant for all θ. At a minimum of T, for physically acceptable solutions, the following conditions must hold:

$$T_k = 0, \quad T_\theta = 0. \tag{2.8}$$

In Hall (1984) these conditions are satisfied on the real θ-axis, but clearly complex values of θ are also permissible. For the narrow spherical annulus problem studied by Walton (1978), Soward & Jones found that, at the minimum value of T on $\theta = 0$, although $T_k = 0$, T_θ did not vanish, and the resulting amplitude equation is of the Airy type, which does not lead to physically realistic solutions.

If we are to obtain valid solutions that describe the vortex amplitudes at the critical Taylor number, we must locate the saddle points of T at which the conditions of (2.8) are met. Since, for our problem, this does not occur for real θ we must analytically continue the solution into the complex θ-plane. Suppose that such a point exists, say at $\theta = \theta_0$, then the solution found in the neighbourhood of this point provides both the inner solution, that removes the singularity at the outer bend, $\theta = 0$, and also a set of asymptotic solutions that match with those valid away from the saddle point, and which describe the flow field as $\beta \to 0$, for all real θ.

In addition, we note that due to the symmetry of the problem, we shall be considering only modes that are symmetric with respect to θ. Thus,

$$\mathrm{Re}(\theta_0) = 0, \quad \mathrm{Im}(K_0) = 0. \tag{2.9}$$

An inner variable Θ, defined by

$$\theta - \theta_0 = \beta^{1/2}\Theta, \tag{2.10}$$

is introduced and we note that Θ is of $O(1)$.

We also introduce the following notation for the perturbation vector, $\hat{\boldsymbol{q}}$:

$$\hat{\boldsymbol{q}} = \left(\hat{p}, \frac{\partial \hat{v}}{\partial \eta}, \frac{\partial \hat{w}}{\partial \eta}, \hat{u}, \hat{v}, \hat{w}\right). \tag{2.11}$$

The perturbation vector $\hat{\boldsymbol{q}}$ is then expanded in powers of $\beta^{1/2}$ as

$$\hat{\boldsymbol{q}} = d_1(\Theta)\hat{\boldsymbol{q}}_1 E + d_2(\Theta)\hat{\boldsymbol{q}}_2 E\beta^{1/2} + d_3(\Theta)\hat{\boldsymbol{q}}_3 E\beta + \cdots + \text{c.c.}, \tag{2.12}$$

where $E = \exp[\mathrm{i}k_0\Theta/\beta^{1/2}]$ and c.c. denotes the complex conjugate. The $\hat{\boldsymbol{q}}_i, i = 1, 2, \ldots$ are defined in a similar manner to $\hat{\boldsymbol{q}}$. Having introduced a disturbance, $\hat{\boldsymbol{q}}$, into the basic flow within the Stokes layer, we describe the total flow as

$$(u, v, w, p) = (u_B, v_B, w_B, p_B) + \epsilon_1(\hat{u}, \hat{v}, \hat{w}, \hat{p}),$$

where ϵ_1 is a small vanishing parameter, and the subscript B is, once again, used to denote quantities of the unperturbed, basic flow field.

It is now a straighforward procedure to substitute the total flow field into the governing equations and linearize with respect to ϵ_1. The leading-order problem for $\hat{\boldsymbol{q}}_1$, reduces to

$$\hat{\mathbf{C}}_0(\hat{\boldsymbol{q}}_1) = 0, \tag{2.13}$$

where $\hat{\mathbf{C}}_0$ is the operator defined by

$$\hat{\mathbf{C}}_0 = \mathbf{I}\frac{\partial}{\partial \eta} - \hat{\mathbf{A}}_0 + \mathbf{B}\frac{\partial}{\partial \tau}. \tag{2.14}$$

In the above, $\hat{\mathbf{A}}_0$ and $\mathbf{B}$ are 6×6 matrices not given here.

At $O(\beta^{1/2})$, $\hat{\boldsymbol{q}}_2$ satisfies,

$$d_2\left[\mathbf{I}\frac{\partial \hat{\boldsymbol{q}}_2}{\partial \eta} - \hat{\mathbf{A}}_0\hat{\boldsymbol{q}}_2 + \mathbf{B}\frac{\partial \hat{\boldsymbol{q}}_2}{\partial \tau}\right] = d_{1\Theta}\mathbf{M}_1\hat{\boldsymbol{q}}_1 + \Theta d_1\mathbf{M}_2\hat{\boldsymbol{q}}_1 + d_1\mathbf{M}_3\hat{\boldsymbol{q}}_1. \tag{2.15}$$

Here $\boldsymbol{M}_i, i = 1, 2, 3$, are 6×6 matrices not given here. The solution to (2.15) is of the form

$$d_2\hat{\boldsymbol{q}}_2 = -\mathrm{i}d_{1\Theta}\hat{\boldsymbol{q}}_2^{(1)} + \mathrm{i}\Theta d_1\hat{\boldsymbol{q}}_2^{(2)} + d_1\hat{\boldsymbol{q}}_2^{(3)}, \tag{2.16}$$

where $\hat{\boldsymbol{q}}_2^{(i)} i = 1, 2, 3$ are evaluated using the following system of equations:

$$\left.\begin{aligned} \hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_2^{(1)}) &= -\hat{\boldsymbol{C}}_{k_0 0}(\hat{\boldsymbol{q}}_1), \\ \hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_2^{(2)}) &= \mathrm{i}\hat{\boldsymbol{C}}_{\theta_0 0}(\hat{\boldsymbol{q}}_1), \\ \hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_2^{(3)}) &= \mathrm{i}\hat{\boldsymbol{C}}_{k_0 0}(\hat{\boldsymbol{q}}_{1\Theta}). \end{aligned}\right\} \tag{2.17}$$

In (2.17), the subscripts k_0, θ_0 denote partial differentiation with respect to k_0, θ_0 respectively, and zero represents evaluation at $\theta = \theta_0$.

The evolution equation for the leading-order amplitude function d_1 is determined by imposing a solvability condition on the differential system obtained at $O(\beta)$. The required condition is

$$I_M d_{1\Theta\Theta} + I_N\Theta d_{1\Theta} + I_P d_1 + I_Q d_{1\Theta} + I_R\Theta d_1 + I_S\Theta^2 d_1 = 0, \tag{2.18}$$

where the coefficients $I_N \rightarrow I_S$ are double integrals over η- and τ-space, for example,

$$I_M = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} V^T(\boldsymbol{M}^{(0)}\hat{\boldsymbol{q}}_1 + \boldsymbol{M}^{(1)}\hat{\boldsymbol{q}}_2^{(1)})\mathrm{d}\tau\mathrm{d}\eta. \tag{2.19}$$

3. Formulation of the nonlinear stability problem

We now go on to describe how nonlinear effects alter the linear stability analysis outlined in § 2.2. The terms *fundamental*, *mean* and *first-harmonic* refer to the dependence of the disturbance on the θ-coordinate. We recall that the θ-dependent amplitude of the disturbance satisfies (2.18) and that exponentially decaying solutions exist for only certain values of the Taylor number, T, that is those satisfying the conditions (2.8). We scale the amplitude of the disturbance in such a way as to introduce nonlinear effects which modify the linear amplitude equation. In order to retain the linear structure of (2.18), we expand the Taylor number as

$$T = T_{0C} + \beta T_1, \tag{3.1}$$

where T_{0C} is the critical value of T_0. We first assume that the velocity field perturbation is $O(\beta^\delta)$. At $O(\beta^{2\delta})$, the fundamental interacts with itself, through the nonlinear terms of the Navier–Stokes equations, to produce first-harmonic and mean flow correction terms. These then interact with the fundamental, reinforcing it at $O(\beta^{3\delta})$. Thus for (2.18) to be modified by nonlinear effects we must choose $\delta = \frac{1}{2}$.

We now outline the construction of the inner solution following the method of Papageorgiou (1987), described briefly in § 2.2. We define an $O(1)$ inner variable Θ by

$$\theta - \theta_0 = \Theta\beta^{1/2}, \tag{3.2}$$

where $\theta = \theta_0$ is the value of θ for which the condition (2.8) are satisfied; k_0 and T_0

are similarly defined. The disturbance quantities u, v, w and p are then expanded as

$$\begin{aligned}\tilde{u} = {} & Ed_{11}u_{11}\beta^{1/2} + [Ed_{21}u_{21} + E^2 d_{22}u_{22} + d_{20}u_{20}]\beta \\ & + [Ed_{31}u_{31} + E^2 d_{32}u_{32} + E^3 d_{33}u_{33} + d_{30}u_{30}]\beta^{3/2} \\ & + \cdots + \text{c.c.},\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\tilde{v} = {} & Ed_{11}u_{11}\beta^{1/2} + [Ed_{21}v_{21} + E^2 d_{22}v_{22} + d_{20}v_{20}]\beta \\ & + [Ed_{31}v_{31} + E^2 d_{32}v_{32} + E^3 d_{33}v_{33} + d_{30}v_{30}]\beta^{3/2} \\ & + \cdots + \text{c.c.},\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\tilde{w} = {} & Ed_{11}w_{11}\beta^{1/2} + [Ed_{21}w_{21} + E^2 d_{22}w_{22} + d_{20}w_{20}]\beta \\ & + [Ed_{31}w_{31} + E^2 d_{32}w_{32} + E^3 d_{33}w_{33} + d_{30}w_{30}]\beta^{3/2} \\ & + \cdots + \text{c.c.},\end{aligned} \tag{3.3c}$$

$$\begin{aligned}\tilde{p} = {} & Ed_{11}p_{11}\beta^{1/2} + [Ed_{21}p_{21} + E^2 d_{22}p_{22} + d_{20}p_{20}]\beta \\ & + [Ed_{31}p_{31} + E^2 d_{32}p_{32} + E^3 d_{33}p_{33} + d_{30}p_{30}]\beta^{3/2} \\ & + \cdots + \text{c.c.}\end{aligned} \tag{3.3d}$$

The coefficients in the above expansions depend only on the variable τ, Θ and η, c.c. denotes the complex conjugate and E is given by

$$E = \exp[\mathrm{i}k_0\Theta/\beta^{1/2}] \tag{3.4}$$

Hence, the total flow may be expressed as

$$(u, v, w, p) = (u_B, v_B, w_B, p_B) + (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}), \tag{3.5}$$

where the subscript B denotes the basic flow. At this point we introduce the perturbation vector $\boldsymbol{q}_{ij}$, defined as

$$\boldsymbol{q}_{ij} = \left[p_{ij}, \frac{\partial v_{ij}}{\partial \eta}, \frac{\partial w_{ij}}{\partial \eta}, u_{ij}, v_{ij}, w_{ij}\right]. \tag{3.6}$$

We proceed by substituting (3.5) into the governing equations and successively equate like powers of $\beta^{1/2}$, so that at $O(\beta^{1/2})$ we find that the fundamental, $\boldsymbol{q}_{11}$, satisfies the linear stability problem, with θ, k and T evaluated at the saddle point of T,

$$\hat{\boldsymbol{C}}_0(\boldsymbol{q}_{11}) = 0, \tag{3.7}$$

where $\hat{\boldsymbol{C}}_0$ is the operator defined by,

$$\hat{\boldsymbol{C}}_0 = \boldsymbol{I}\frac{\partial}{\partial \eta} - \hat{\boldsymbol{A}}_0 + \hat{\boldsymbol{B}}\frac{\partial}{\partial \tau}. \tag{3.8}$$

$\hat{\boldsymbol{A}}_0$ is the matrix $\boldsymbol{A}$ evaluated at the saddle point of T

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\frac{1}{2}\mathrm{i}k & 0 & \frac{1}{2}k^2 + \mathrm{i}kT_0v_{B0} & 0 & -2w_{B0}\cos\theta \\ 2\mathrm{i}k & 0 & 0 & -2T_0v_{B0\eta} & k^2 + 2\mathrm{i}kT_0v_{B0} & 4w_{B0}\sin\theta \\ 0 & 0 & 0 & -2T_0w_{B0\eta} & 0 & k^2 + 2\mathrm{i}kT_0v_{B0} \\ 0 & 0 & 0 & 0 & \mathrm{i}k & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.9}$$

and $\boldsymbol{B}$ is the 6×6 matrix

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.10}$$

At $O(\beta)$, the fundamental, $\boldsymbol{q}_{21}$, satisfies

$$d_{21}\hat{\boldsymbol{C}}_0(\boldsymbol{q}_{21}) = d_{11}\boldsymbol{M}_1\boldsymbol{q}_{11\Theta} + d_{11}\Theta\boldsymbol{M}_2\boldsymbol{q}_{11} + d_{11\Theta}\boldsymbol{M}_1\boldsymbol{q}_{11} \tag{3.11}$$

where the matrices $\boldsymbol{M}_i, i = 1, 2$, are related to the operator $\hat{\boldsymbol{C}}_0$ through the following equations:

$$\boldsymbol{M}_1 = \mathrm{i}\frac{\partial\hat{\boldsymbol{C}}_0}{\partial k_0}, \quad \boldsymbol{M}_2 = -\frac{\partial\hat{\boldsymbol{C}}_0}{\partial\theta_0}, \tag{3.12a,b}$$

and we may obtain a solution to (3.11) by writing

$$d_{21}\hat{\boldsymbol{q}}_{21} = -\mathrm{i}d_{11\Theta}\hat{\boldsymbol{q}}_{21}^{(1)} + \mathrm{i}\Theta d_{11}\hat{\boldsymbol{q}}_{21}^{(2)} + d_{11}\hat{\boldsymbol{q}}_{21}^{(3)}, \tag{3.13}$$

where $\hat{\boldsymbol{q}}_{21}^{(i)}, i = 1, 2, 3$, are to be determined. Substitution of (3.13) into (3.11) and equating coefficients of $d_{11\Theta}, \Theta d_{11}$ and d_{11} yields the following equations for $\hat{\boldsymbol{q}}_{21}^{(1)}, \hat{\boldsymbol{q}}_{21}^{(2)}$ and $\hat{\boldsymbol{q}}_{21}^{(3)}$:

$$\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{21}^{(1)}) = -\hat{\boldsymbol{C}}_{k_0 0}(\hat{\boldsymbol{q}}_{11}), \tag{3.14a}$$

$$\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{21}^{(2)}) = \mathrm{i}\hat{\boldsymbol{C}}_{\theta_0 0}(\hat{\boldsymbol{q}}_{11}), \tag{3.14b}$$

$$\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{21}^{(3)}) = \mathrm{i}\hat{\boldsymbol{C}}_{k_0 0}(\hat{\boldsymbol{q}}_{11\Theta}). \tag{3.14c}$$

In (3.14*a*–*c*), the subscripts k_0, θ_0 denote partial differentiation with respect to k_0, θ_0 respectively, and the zero subscript denotes evaluation at the saddle point of T, that is where. In addition to the fundamental mode $\boldsymbol{q}_{21}$, first harmonic $\boldsymbol{q}_{22}$ and mean flow correction $\boldsymbol{q}_{20}$ terms are also generated. After some manipulation, we find that these maybe expressed as

$$d_{22}\boldsymbol{q}_{22} = d_{11}^2\hat{\boldsymbol{q}}_{22}, \quad d_{20}\boldsymbol{q}_{20} = d_{11}d_{11}\hat{\boldsymbol{q}}_{20}. \tag{3.15a,b}$$

In addition, $\hat{\boldsymbol{q}}_{22}, \hat{\boldsymbol{q}}_{20}$ satisfy

$$\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{22}) = \boldsymbol{D}, \tag{3.16}$$

$$\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{20}) = \boldsymbol{D}. \tag{3.17}$$

In the above equations of $\boldsymbol{q}_{22}$ and $\boldsymbol{q}_{20}$, $\boldsymbol{D}$ and $\hat{\boldsymbol{D}}$ represent the terms arising form the interaction of the leading-order fundamental with itself.

The amplitude function d_{11} is obtained by imposing a solvability condition on the differential system for the fundamental $\boldsymbol{q}_{31}$ at $O(\beta^{3/2})$,

$$\begin{aligned} d_{31}\hat{\boldsymbol{C}}_0(\hat{\boldsymbol{q}}_{31}) = {} & d_{11\Theta\Theta}[\boldsymbol{M}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{M}^{(1)}\hat{\boldsymbol{q}}_{21}^{(1)}] \\ & + d_{11\Theta}\Theta[\boldsymbol{N}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{N}^{(1)}\hat{\boldsymbol{q}}_{21}^{(1)} + \boldsymbol{N}^{(2)}\hat{\boldsymbol{q}}_{21}^{(2)}] + d_{11}[\boldsymbol{P}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{P}^{(2)}\hat{\boldsymbol{q}}_{21}^{(2)} + \boldsymbol{P}^{(3)}\hat{\boldsymbol{q}}_{21}^{(3)}] \\ & + d_{11\Theta}[\boldsymbol{Q}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{Q}^{(1)}\hat{\boldsymbol{q}}_{21}^{(1)} + \boldsymbol{Q}^{(3)}\hat{\boldsymbol{q}}_{21}^{(3)} + \Theta d_{11}[\boldsymbol{R}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{R}^{(2)}\hat{\boldsymbol{q}}_{21}^{(2)} + \boldsymbol{R}^{(3)}\hat{\boldsymbol{q}}_{21}^{(3)}] \\ & + \Theta^2 d_{11}[\boldsymbol{S}^{(0)}\hat{\boldsymbol{q}}_{11} + \boldsymbol{S}^{(2)}\hat{\boldsymbol{q}}_{21}^{(2)}] + d_{11}|d_{11}^2|\boldsymbol{T}^{(0)}\hat{\boldsymbol{q}}_{11}, \end{aligned} \tag{3.18}$$

where $\boldsymbol{M}^{(i)}, \boldsymbol{N}^{(i)}, \boldsymbol{P}^{(i)}, \boldsymbol{Q}^{(i)}, \boldsymbol{R}^{(i)}, \boldsymbol{S}^{(i)}, i = 0,1,2,3$, are 6×6 matrices not given here, and $\boldsymbol{T}^{(0)}$ contains the terms arising from the interaction of the leading-order fundamental with the first-harmonic and mean flow correction terms. By considering the partial differential system adjoint to (3.7), we find that a solution to the above equation exists if

$$d_{11\Theta\Theta} I_M + \Theta d_{11\Theta} I_N + d_{11} I_P + d_{11\Theta} I_Q + \Theta d_{11} I_R + \Theta^2 d_{11} I_S + d_{11}|d_{11}^2| I_T = 0, \quad (3.19)$$

where I_M etc. are double integrals over η- and τ-space. For example,

$$I_M = \int_{\eta=0}^{\infty} \int_{\tau=0}^{2\pi} \boldsymbol{V}^T [\boldsymbol{M}^{(0)} \hat{\boldsymbol{q}}_{11} + \boldsymbol{M}^{(1)} \hat{\boldsymbol{q}}_{21}^{(1)}] \mathrm{d}\tau \mathrm{d}\eta, \quad (3.20)$$

and $\boldsymbol{V}$ satisfies the adjoint differential system,

$$\hat{\boldsymbol{C}}_0^{\dagger}(\boldsymbol{V}) = 0 \quad (3.21)$$

$$V_1 = V_2 = V_3 = 0 \quad \text{at} \quad \eta = 0 \quad (3.22)$$

where $\hat{\boldsymbol{C}}_0^{\dagger}$ is the operator

$$\hat{\boldsymbol{C}}_0^{\dagger} = \frac{\partial}{\partial \eta} + \hat{\boldsymbol{A}}_0^T + \boldsymbol{B}^T \frac{\partial}{\partial \tau}. \quad (3.23)$$

The amplitude equation (3.19) must be solved subject to the boundary conditions

$$d_{11} \to 0 \quad \text{as} \quad |\Theta| \to \infty. \quad (3.24)$$

4. Numerical solution and results

In this section, we describe the numerical procedure used to solve the leading-order eigenvalue problem subject to boundary conditions of no-slip at the pipe wall and decay of the disturbance at infinity.

The leading-order problem must be solved subject to conditions on k_0 and θ_0. The eigenrelation is solved for fixed values of k and θ until the value of T_0 corresponding to physically acceptable solutions is located.

On the basis of Floquet theory, since the basic flow is periodic in time, we assume that the disturbances are also periodic and carry out a Fourier expansion in time for the perturbation quantities. For neutrally stable solutions this takes the form

$$\boldsymbol{q} = G \sum_{-\infty}^{\infty} \boldsymbol{q}^n \mathrm{e}^{\mathrm{i}n\tau} + \text{c.c.}, \quad (4.1)$$

where G is a constant and $\boldsymbol{q}^n$ are functions of η alone. After some manipulation, the equations for the leading-order problem may be reduced to a pair of coupled partial differential equations for u and w,

$$\left(\frac{\partial^2}{\partial \eta^2} - k^2 - 2\frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2}{\partial \eta^2} - k^2\right) u_{11} + 2\mathrm{i}k T_0 (k^2 v_{B0} + v_{B0\eta\eta}) u_{11} - 2\mathrm{i}k T_0 v_{B0} u_{11\eta\eta}$$
$$-4k^2 w_{B0} w_{11} \cos\theta_0 - 4\mathrm{i}k \sin\theta_0 (w_{B0\eta} w_{11} + w_{B0} w_{11\eta}) = 0, \quad (4.2a)$$

$$\left(\frac{\partial^2}{\partial \eta^2} - k^2 - 2\frac{\partial}{\partial \tau}\right) w_{11} + 2T_0 w_{B0\eta} u_{11} - 2\mathrm{i} T_0 v_{B0} w_{11} = 0. \quad (4.2b)$$

Substitution of (4.1) into the above system would lead to an infinite set of coupled ordinary differential equations at successive powers of $\mathrm{e}^{\mathrm{i}n\tau}$. In order to reduce this to a more tractable system of equations, we set $u_n = w_n = 0$ for $|n| > M$. In addition, we replace ∞ by η_∞, where η_∞ and M are chosen to give the degree of accuracy required.

For larger η, (4.2*a,b*) reduces to

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) u_{11} - \tfrac{1}{2}\mathrm{i}kT_0k^2\sin\theta_0 u_{11} + \tfrac{1}{2}\mathrm{i}kT_0\sin\theta_0 u_{11\eta\eta}$$
$$-4k^2\sin\tau w_{11}\cos\theta_0 - 4\mathrm{i}k\sin\theta_0\sin\tau w_{11\eta} = 0, \qquad (4.3a)$$

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right) w_{11} + \tfrac{1}{2}\mathrm{i}kT_0\sin\theta_0 w_{11} = 0. \qquad (4.3b)$$

The above system has three independent solutions with the correct behaviour as $\eta \to \infty$. These may be used to integrate (4.2*a,b*) from $\eta = \eta_\infty$ to $\eta = 0$, in this instance using a fourth-order Runge–Kutta scheme with h steps in the interval of integration, to provide $3(2M + 1)$ solutions. Combining the independent solutions of the reduced system at $\eta = 0$ we may satisfy all but one of the boundary conditions, and the remaining condition is met if $T(k, \theta)$ is an eigenvalue of the system.

Our calculations, for $M = 6$, $\eta_\infty = 12.5$ and $h = 251$ gave results of $T_0 = 10.7301, \theta_0 = 0.3525\mathrm{i}$ and $k_0 = 0.5313$, comparing favourably with those given by Papageorgiou (1987).

The eigenfunctions corresponding to (4.2*a,b*) were found to have the property that

$$u_{11}^n = 0, \qquad n \text{ even}, \qquad (4.4a)$$

$$w_{11}^n = 0, \qquad n \text{ odd}. \qquad (4.4b)$$

The functions, u_{11}^{-1} and w_{11}^0 evaluated at the critical values of k_0, θ_0 and T_0 are illustrated in figure 2, together with the eigenfunctions for u_{11}^{-3}. The solutions were normalized such that the first derivative of $w_{11}^{(0)}$ was set to unity.

The solution to the adjoint solution was calculated in the same manner and used as a check on the values found above, since the eigenvalues of the adjoint system are identical with those of (4.2*a,b*). The functions $\boldsymbol{V}_1^{-1}$ and $\boldsymbol{V}_3^0$ of the adjoint system, plotted as functions of η and at the critical value of T_0, are illustrated in figure 3. The adjoint eigenfunctions are such that

$$\boldsymbol{V}_1^n = 0, \qquad n \text{ even}, \qquad (4.5a)$$

$$\boldsymbol{V}_3^n = 0, \qquad n \text{ odd}. \qquad (4.5b)$$

In addition solutions for the fundamental, mean flow correction and first harmonic at $O(\beta)$ were calculated in a similar fashion and hence the integral coefficients of the amplitude equations (2.18), (3.19) could be calculated, in this case using Simpson's rule.

Before proceeding we first simplify the linear amplitude equation, so that the solution may be expressed in terms of known functions. Using the transformations

$$d_1 = \mathrm{e}^{\left(-\frac{\alpha^2}{2}(\zeta-\gamma)^2\right)}\mathrm{e}^{(-\lambda\zeta)}Z_l, \qquad (4.6a)$$

$$\zeta = \Theta + \gamma, \qquad (4.6b)$$

we may re-write (2.18) as

$$Z_l'' + \mu\left(\frac{H + C_{PT}T_1}{\mu} - \zeta^2\right) Z_l = 0, \qquad (4.7)$$

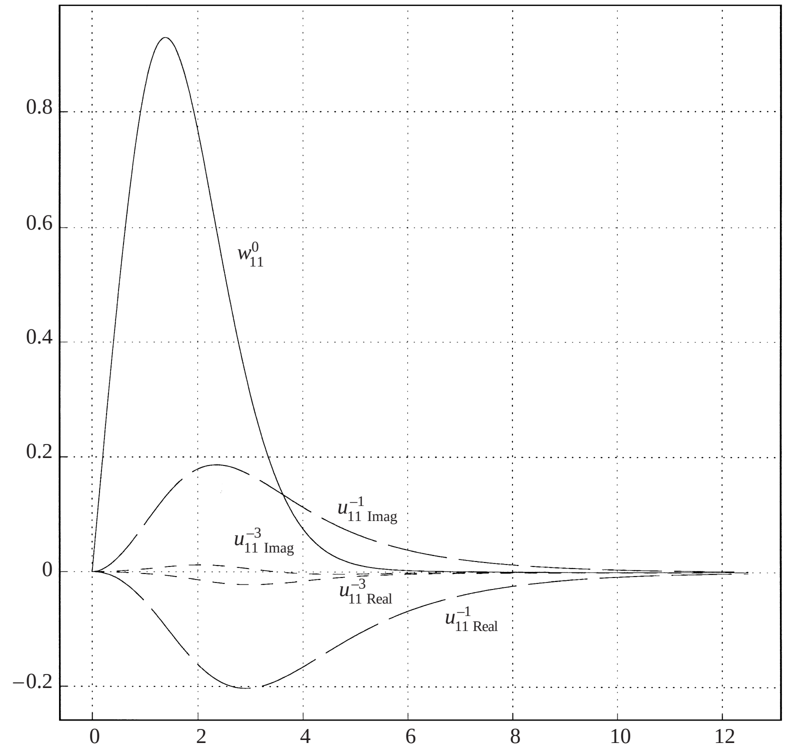

FIGURE 2. The eigenfunctions u^{-1}_{11}, u^{-3}_{11} and w^0_{11}, plotted as a function of η for the critical value of T_0.

where

$$\mu = \left(\frac{\alpha^4}{2} + \frac{I_S}{I_M}\right), \tag{4.8a}$$

$$H = \lambda^2 - \gamma^2\mu + \frac{I_P}{I_M} - \alpha^2 - \gamma\left(\frac{I_R}{I_M} - \frac{I_Q}{I_M}\alpha^2\right) - \lambda\frac{I_S}{I_M}, \tag{4.8b}$$

$$C_{PT} = \frac{I_{PT}}{I_M}, \tag{4.8c}$$

In the above transformations, α, λ and γ are suitably chosen constants, depending on the integral coefficients of (3.19). Since we have assumed in (4.1) that the coefficients of the Fourier expansion depend only on η, we find that λ and γ are both equal to zero. The amplitude function d_1, and hence, Z_l, must decay as $|\Theta| \to \infty$, and hence the solutions to (4.7) are

$$Z_l(\zeta) = Z_{ln}(\zeta) = U_n\left(-\left(n + \tfrac{1}{2}\right), 2^{1/2}\mu^{1/4}\zeta\right), \tag{4.9}$$

where U_n is the nth parabolic cylinder function, and the corresponding value of T_1 is

$$T_1 = T_{1n} = \frac{2\mu^{1/2}(n + \frac{1}{2}) - H}{C_{PT}}. \tag{4.10}$$

The functions $Z_{ln}(\zeta)$ have $n-1$ zeros for $\zeta \in (-\infty, +\infty)$ and depending on the odd/even nature of n, are odd or even in ζ. All the functions tend to zero like

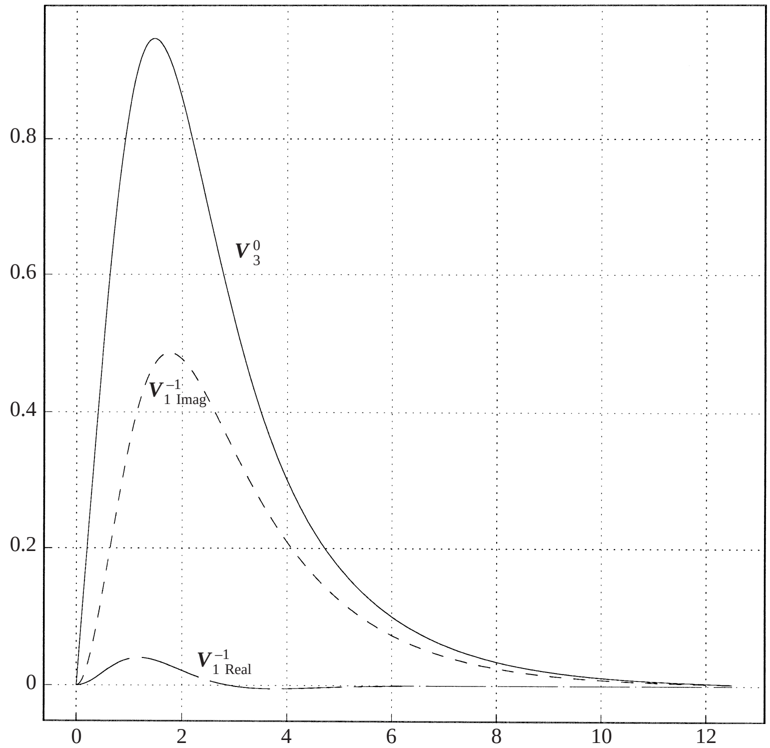

FIGURE 3. The eigenfunctions V_1^{-1} and V_3^0 for the adjoint problem corresponding to the critical Taylor number, T_0.

$\exp(-\mu^{1/2}\zeta^2/2)$ and the least-stable mode corresponds to the $n = 0$ case, when

$$Z_{l0} = \exp\left(-\mu^{1/2}\frac{\zeta^2}{2}\right),$$

and

$$T_1 = T_{1C} = \frac{\mu^{1/2} - H}{C_{PT}}. \tag{4.11}$$

Our calculations showed that $T_{1C} = 3.005$.

Returning to the nonlinear amplitude equation (3.19), using the transformation (4.6*a*,*b*) this maybe re-written in the following form:

$$Z'' + \mu\left(\frac{H + C_{PT}T_1}{\mu} - \zeta^2\right)Z = -C_T Z|Z|^2 F(\zeta), \tag{4.12}$$

where

$$F(\zeta) = |\mathrm{e}^{-\lambda\zeta}\mathrm{e}^{\frac{\alpha^2}{2}(\zeta-\gamma)^2}|^2, \tag{4.13a}$$

$$C_T = \frac{I_T}{I_M}, \tag{4.13b}$$

and H, μ, C_{PT} are given by (4.8*a*–*c*).

In order to progress analytically, and provide some comparison with the numerical

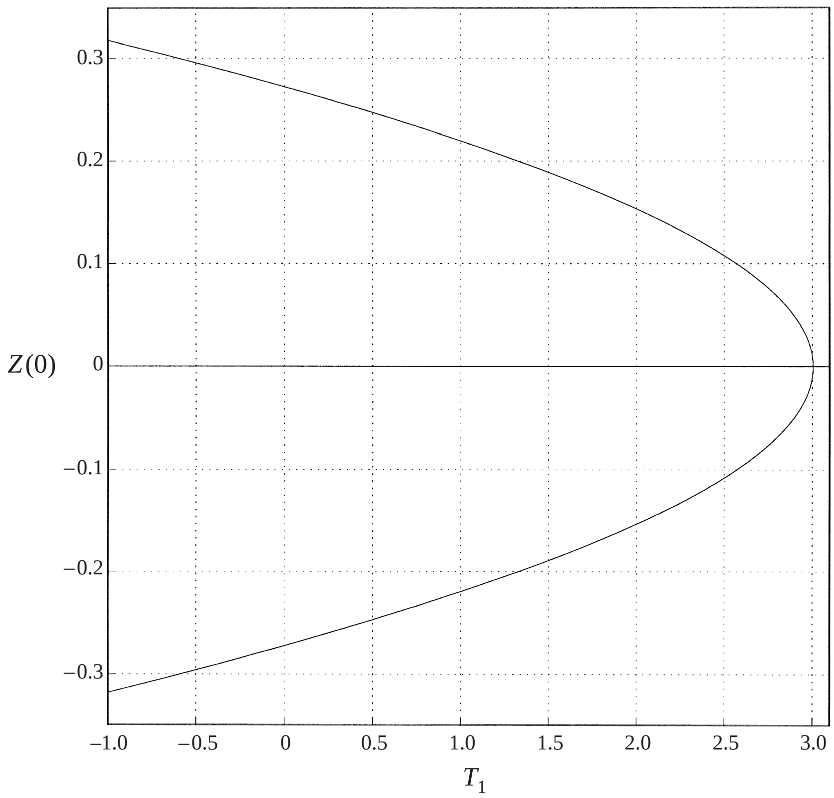

FIGURE 4. The numerically calculated bifurcating solution of the normalized nonlinear amplitude equation (4.12).

solution to (4.12), we expand Z and T_1 in terms of some small parameter $\hat{\epsilon}$,

$$Z = \hat{\epsilon}^{1/2} Z_1 + \hat{\epsilon}^{3/2} Z_3 + \cdots,$$

$$T_1 = T_C + \hat{\epsilon} T_P + \cdots,$$

substitute these into (4.12) and successively equate like powers of $\hat{\epsilon}^{1/2}$. At leading order, Z_1 satisfies the linear amplitude equation (4.7) and hence

$$Z_1 = \phi Z_l,$$

where ϕ is an arbitrary constant determined at higher order. In addition, $T_C = T_{1C}$, where T_{1C} is given by (4.11). At $O(\hat{\epsilon}^{3/2})$, we find that Z_3 satisfies

$$Z_3'' + \mu \left(\frac{H + C_{PT} T_C}{\mu} - \zeta^2 \right) Z_3 = -C_T \phi Z_l |\phi Z_l|^2 F(\zeta) - C_{PT} T_P \phi Z_l. \tag{4.14}$$

In order for solutions to the above equation to exist, a solvability condition must be satisfied. By this we mean that the right-hand side of the differential equation must be orthogonal to the solution of the adjoint problem to (4.7). It should be noted that (4.7) is self-adjoint. After some manipulation, the solvability condition reduces to

$$|\phi|^2 = -\frac{C_{PT}}{C_T} \frac{J_1}{J_2} T_P \tag{4.15}$$

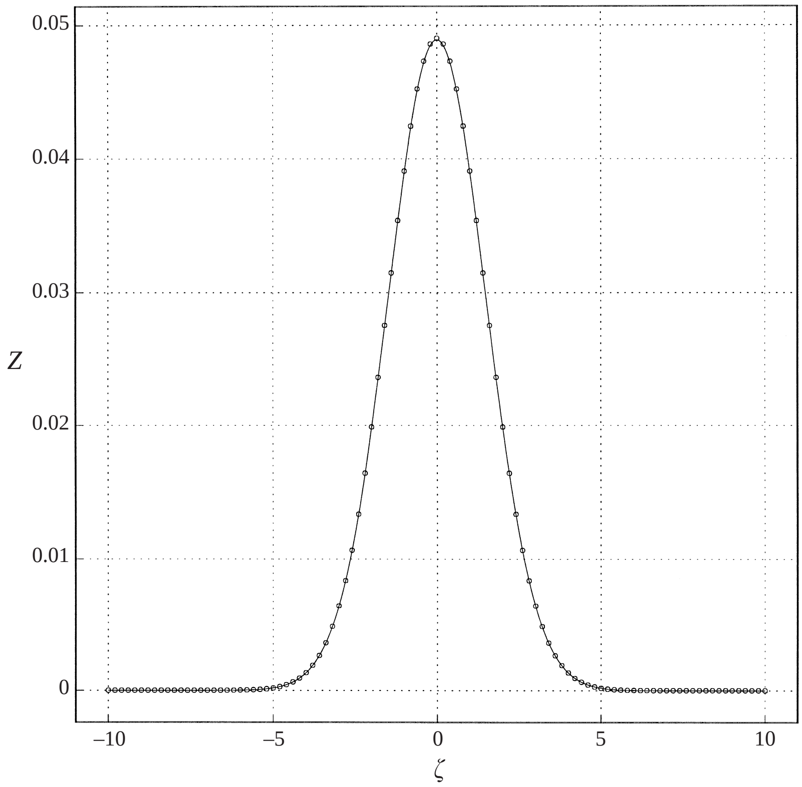

FIGURE 5. Comparison of solutions for $T_1 = 2.9$. The solid line represents the solution of the full problem (4.12) and the ∘ symbol the approximate solution (4.14).

where

$$J_1 = \int_{-\infty}^{\infty} |Z_l|^2 \mathrm{d}\zeta, \quad J_2 = \int_{-\infty}^{\infty} |Z_l|^4 F(\zeta) \mathrm{d}\zeta.$$

If we assume that $|T_1 - T_C| \ll 1$ then (4.15) maybe be written in the form

$$|Z|^2 \sim -\frac{C_{PT}}{C_T}(T_1 - T_C)|Z_l|^2 \frac{J_1}{J_2}. \tag{4.16}$$

In this analysis, only the quantities C_{PT} and C_T are needed to calculate the nature of the instability, and since C_{PT}/C_T was found to be positive, finite-amplitude solutions of (4.12) only exist locally near $T_1 = T_{1C}$ for $T_1 < T_{1C}$. Thus the solutions to (4.12) will bifurcate subcritically from the eigenvalues of the linear problem. The subcritical nature of the bifurcation was confirmed by numerically integrating (4.12) using a shooting procedure. We note that, since the differential operator in (4.12) is even in ζ, and $F(\zeta)$ is also an even function for $\lambda = 0$, we can expect that the solution $Z(\zeta)$ is either even or odd in ζ, depending on the conditions applied, with even solutions corresponding to $Z'(0) = 0$ and odd solutions to $Z'(0^+) = Z'(0^-)$, and in this case the former condition was applied. The results are shown in figure 4, where we have plotted the amplitude of the first mode, evaluated at $\zeta = 0$, as a function of T_1. It is possible that higher-order nonlinear effects may reverse this result, producing supercritical equilibrium solutions; however, higher-order calculations would be required in order to substantiate/disprove this conjecture.

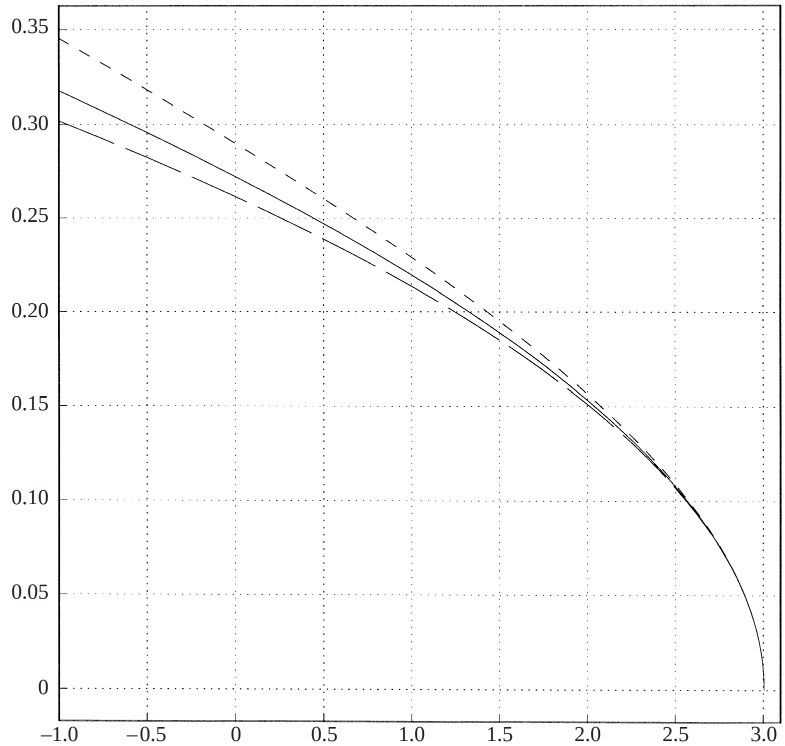

FIGURE 6. Comparison of $|Z(0)|^2$ from (4.12) (solid line) (4.14) (small-dashed line) and (4.16) (dashed line).

Subcritical solutions are known to be unstable, and by allowing the amplitude function Z to be dependent on some slow time variable this may be shown. Thus the fully nonlinear problem must be solved numerically in order to find out how the flow develops.

In figure 5 we compare the results obtained from numerically integrating (4.12) (solid line) with those from the expansion procedure carried out in the neighbourhood of T_{1C}. We took $T_1 = 2.9$, and as illustrated the results were found to be almost identical.

The results obtained by numerically integrating the nonlinear amplitude equation (4.12) (solid line), are compared with those obtained from the expansion about T_{1C} (small-dashed line) and the approximate value for $|Z|^2$ from (4.16) (dashed line) in figure 6. As expected the results from the two approximate methods are in good agreement with those from the numerical integration in the neighbourhood of T_{1C}.

5. Conclusions

We have obtained the equation governing the nonlinear evolution of the leading-order vortex amplitude function $d_{11}(\Theta)$. As expected, the linear terms of this equation are of the same form as those of the equivalent equation found from the linear stability analysis. Indeed, we would also expect that the coefficients of the linear terms, $I_M, I_N, I_P, I_Q, I_R, I_S$, will be identical to those for the linear evolution equation, found by Papageorgiou (1987), taking account of the differences in notation.

The results of our numerical calculations predict that the instability is subcritical in nature, and thus we presume that close to the critical Taylor number, sufficiently large perturbations to the basic state will grow. These disturbances might tend to some equilibrium, due to stabilization by higher-order nonlinear effects, and thus some form of steady state reached. Alternatively, even if no stabilization takes place, due to the localized nature of the instability, we might expect that some periodicity along the pipe exists. For example, Tollmien–Schlichting waves are subcritical in nature, but can be observed in parallel or nearly parallel flows. The fully nonlinear problem would need to be solved numerically in order to investigate the subcritical nature of the bifurcation, and find the flow into which the disturbance evolves.

In the Introduction, we mentioned the relevance of the model to the study of the fluid mechanics of blood flow in large arteries and in particular the aortic arch. Typically for the canine cardiovascular system, the ascending aorta is 1.5 cm in diameter, the mean velocity is approximately $20\,\mathrm{cm\,s^{-1}}$, $R_s \approx 4000$, $\beta \approx 0.1$, and δ has a value of about 0.2. Our analysis is not inconsistent with these values, assuming that the mechanics of the blood flow are not significantly altered in the limit $\delta \to 0$.

However, one important feature of physiological flows is that in general the pressure gradient has a non-zero mean component, and thus the Dean number, D, must be included as a parameter in the problem. For the canine aorta this has a value of $\approx$2000. The problem under consideration here becomes physiologically viable if we consider the case $\beta \to 0, D \leqslant R_s$ (see Papageorgiou). In this case, the flow field within the Stokes layer, to leading order, is described by Lyne's analysis and the stability of the solution is as described here.

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